College Students' Understanding of Linear Functions: Slope is Slippery

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Abstract

A common obstacle for students in the transition from arithmetic to algebra is developing a conceptual understanding of equations representing functions. Two experiments manipulated isomorphic problems in terms of their solution requirements (computation vs. interpretation) and format to test for understanding of linear functions. Experiment 1 provided problems in a story context, and found that performance on slope comparison problems was low, especially when problems were presented with equations. Experiment 2 tested whether performance on slope comparison problems improves when problem prompts include explicit mathematical terminology rather than just natural language consistent with the problem story. Results suggest that many undergraduate students fail to access the mathematical concept of slope when problem prompts are presented with natural language. Overall, the results suggest that even undergraduate students lack understanding of the slope concept and equations of linear functions, both which are foundational for advanced algebraic thinking.

Keywords: algebraic problem solving, slope, linear functions

Introduction

A common obstacle for students in the transition from arithmetic to algebra is developing a conceptual understanding of equations representing functions. Prior work has demonstrated that students can persist in procedural approaches rather than conceptual approaches (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001); process-based approaches rather than object-based approaches (Kieran, 1992); operational or computational approaches rather than relational approaches (Chesney & McNeil, 2014; Kaput, 2000; McNeil & Alibali, 2005) even after algebra instruction. Although much of the research on the transition from arithmetic to algebraic thinking has focused on younger students (middle school and high school), even college students may continue to experience difficulties and lack a mature understanding of linear functions (Hall et al., 1989; Wollman, 1983).

Prior work has suggested that problem presentation plays an important role in solution success. Mevarech and Stern (1997) found that context can affect performance on linear equation problems that are presented graphically. Importantly, the problems that Mevarech and Stern (1997) administered all pertained to the mathematical concept of slope. They found that both younger students (around 12 year olds) and undergraduates performed better when problems were embedded in sparse contexts than when problems were presented in realistic contexts (e.g. graphs depicting how two hoses fill a pool at different rates). Mevarech and Stern (1997) suggested that presenting problems in realistic contexts may overload the problem solver with extraneous information which obscures the underlying mathematical concept and leads to poor performance.

Other work has also found that presentation format can affect undergraduate performance on solving linear equation problems. Mielicki and Wiley (2016) presented undergraduate students with isomorphic problems pertaining to linear functions either in graphical format or with a set of equations. Problems either required computation of a point on a single line (solve for x, solve for y) or required relational reasoning (comparing slopes or comparing points across several linear functions). A main finding was that slope comparison problems were most difficult for students overall, and that these problems were especially difficult when presented in equation format. This finding conforms to the interpretation that Mevarch and Stern (1997) proposed for their findings, and suggests that students may have difficulty accessing relevant mathematical knowledge when slope problems are presented in equation format. Taken together, these findings of significant difference in performance due to problem presentation highlight specific deficits that many undergraduates possess in their understanding of equations representing linear functions.

Experiment 1

The goal of Experiment 1 was to further test the hypothesis that undergraduate students may lack a mature understanding of linear functions expressed as equations, and particularly the mathematical concept of slope. As in Mielicki and Wiley (2016), students were asked to solve isomorphic problems presented either with graphs or with equations. The problems entailed either computation or interpretation across a set of linear functions. Both problem types featured distinct subtypes: computation problems either entailed solving for x or solving y and interpretation problems either entailed comparing the slopes of three linear functions or comparing y values of three linear functions along some range of x values. Examples of each problem type are shown in Figure 1.

Based on a cognitive task analysis, it was expected that the four subtypes of problems would have different cognitive demands in equation and graph format. Overall, computation problems were expected to have higher solution rates than interpretation problems because the former only require consideration of a single linear function whereas the latter require comparing three functions. Between the two subtypes of computation problems, solving for x was expected to be more demanding than solving for y in equation format, since solving for x requires additional computational steps to isolate the variable (all problems were presented in y = mx + b format). For interpretation problems, the cognitive demands for problems requiring the comparison of slopes should not vary by presentation format if students understand the conceptual meaning of what the quantities in the y = mx + b equation represent. In either format, little calculation is necessary – the solver could easily compare the visual trends of the three lines with a graph, or compare the three values of "m" across equations. In contrast, comparing points might be easier to do in graphical format because in equation format multiple calculations are required to compare y values for three equations. These calculations may introduce additional opportunities for error which are not present when point comparison problems are presented with graphs. It was expected that these differences in computational demand would be reflected in solution accuracy.

Method

Participants

A sample of 32 (21 female) undergraduate students from the University of Illinois at Chicago participated in exchange for course credit. Participants were mostly first year undergraduate students (ages ranging from 17 to 23), and had taken 1 math course on average since starting college. Many participants (91%) reported a science (Biology, Chemistry, Psychology, Engineering) or health-related (Pre-med, Pre-nursing, Kinesiology) major. No participants reported pursuing a mathematics major.

Materials

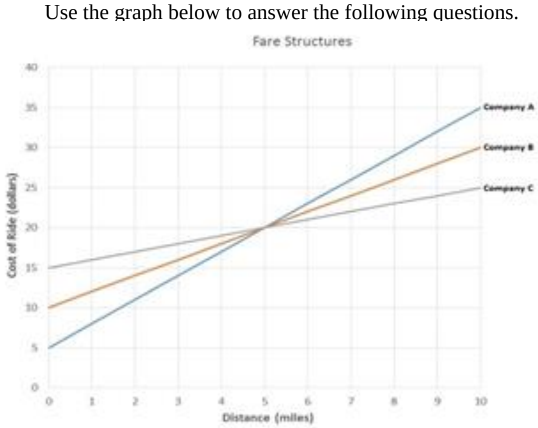
Each participant completed 12 interpretation and 12 computation problems. Computation problems either entailed solving for x or solving for y. Interpretation problems either entailed comparing the slopes or comparing the y values of the three linear functions over a range of x values. Of the 24 problems, half were presented with graphs and half were presented with equations (always in y = mx + b format).

In addition to these manipulations, several other presentation features were either varied in the same way for each participant to reduce monotony, or counterbalanced in order to control for order effects. Each problem was presented individually, but pairs of problems were presented with one format (graphs or equations) and the format for each pair was switched for the second half of the problems so that each participant saw three instances of each problem subtype (solve for x, solve for y, slope comparison, point comparison) in each format (graph, equations). Six configurations of linear functions (one for each block of 4 problems) were used in the same order for each participant. Computation problems requiring solving for x were always paired with slope comparison interpretation problems, and computation problems requiring solving for y were always paired with point comparison interpretation problems. Pairs of problems were alternated, and graph versus equation format was alternated between each pair of problems. In addition, each pair of problems was presented with one of two problem scenarios (i.e., real-world contexts such as comparing cab companies).

Solve for x Problem in Graph Format

Malik is comparing three cab companies. Each company has a different fare structure for charging customers.

Use the graph below to answer the following questions.



If company C charges Malik $25, how many miles did he travel?

Solve for y Problem in Symbolic Format

Bob is participating in a walkathon, and he has gotten three sponsors to donate money to charity for every kilometer he walks. Each sponsor has a different pledge plan for how much money they will donate.

Here are the equations for each sponsor's pledge where y is the amount of money donated in dollars and x is the distance walked in kilometers.

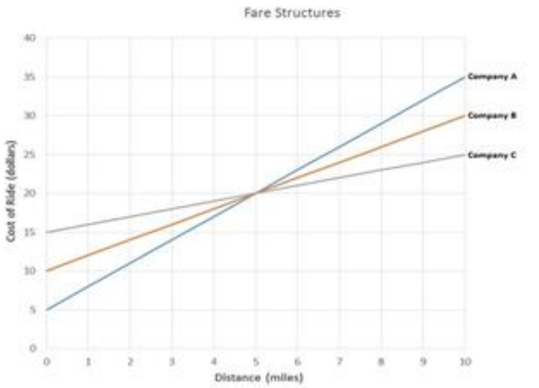
Sponsor A: y = 3x + 5
Sponsor B: y = 2x + 10
Sponsor C: y = x + 15

How much will Sponsor C donate if Bob walks 10 kilometers?

Slope Comparison Problem in Graph Format

Malik is comparing three cab companies. Each company has a different fare structure for charging customers.

Use the graph below to answer the following questions.



Which company has the lowest rate per mile?

Point Comparison Problem in Symbolic Format

Bob is participating in a walkathon, and he has gotten three sponsors to donate money to charity for every kilometer he walks. Each sponsor has a different pledge plan for how much money they will donate.

Here are the equations for each sponsor's pledge where y is the amount of money donated in dollars and x is the distance walked in kilometers.

Sponsor A: $y = 3x + 5$
Sponsor B: $y = 2x + 10$
Sponsor C: $y = x + 15$

Which sponsor will donate the least if Bob walks over 5 kilometers?

Figure 1: Examples of interpretation problems (slope comparison, point comparison) and computation problems (solve for x, solve for y) in either equation or graphical format.

The following were counterbalanced across participants: order of format presentation (graph first versus equation first), order of scenario (scenario A with graph and B with equations versus scenario A with equations and B with graph), order of pair presentation (solve for x/slope comparison pair first versus solve for y/point comparison pair first), and order of problem subtype presentation within pairs of problems (computation first versus interpretation first). This 2x2x2x2 counterbalancing design led to 16 versions, and 2 participants completed each version.

Procedure

All problems were presented one at a time on a computer screen, but participants wrote their responses in an answer booklet that was provided. The answer booklets also included the graphs or equations for each problem so that participants could annotate and interact with the representations as needed. After completing the problems, participants were asked to rephrase a subset of the problems with the prompt, "In your own words, please tell me what you think the problem is asking you to do."

Results

A 2x2 within-subjects ANOVA was conducted with format (graph, equation) and problem type (interpretation, computation) as independent variables and proportion correct as the dependent variable. As shown in Figure 2, the analyses revealed a main effect of representation, $F(1,31) = 27.56$., $p < .001$, $\eta_p^2 = .47$. Participants solved both types of problems more successfully when problems were presented with graphs than when they were presented with equations. The analysis also revealed a main effect of problem type, $F(1,31) = 68.46$, $p < .001$, $\eta_p^2 = .69$. Participants solved computation problems more successfully than interpretation problems, regardless of presentation format. There was a significant interaction, $F(1,31) = 4.93$, $p < .05$, $\eta_p^2 = .14$. Follow up analyses revealed no differences in accuracy for computation problems in different formats, $t(31) = 1.83$, $p = .08$, and higher accuracy for interpretation problems presented in graphical format relative to equation format, $t(31) = 4.51$, $p < .001$.

Additional 2x2 within-subjects ANOVAs were conducted separately for each problem type because the nested subtypes were not orthogonal. For computation problem subtypes, there was no effect of format, $F(1,31) = 3.36$, $p = .08$, or problem subtype, $F(1,31) = 3.14$, $p = .09$, and no interaction, $F < 1$. For interpretation problem subtypes, there was a main effect of format in favor of graphs, $F(1,31) = 20.31$, $p < .001$, $\eta_p^2 = .40$, and a main effect of problem subtype with higher accuracy on point comparison problems than slope comparison problems, $F(1,31) = 43.36$, $p < .001$, $\eta_p^2 = .58$. The interaction was not significant, $F < 1$.

Participants' rephrasing responses for slope comparison problems were coded based on reported solution methods. Most participants (31% for both formats) rephrased the question without providing any indication of a solution method. Some participants referenced the representation, but did not provide a solution method (6% for graphs, 9% for equations), and others mentioned a method but did not provide enough information to determine whether the method was correct or not (6% for graphs, 3% for equations). The remaining responses were coded as either incorrect solution method, or correct solution method/reference to the underlying concept (slope). For slope comparison problems presented with equations, 38% of participants reported an incorrect solution method relative to 25% of participants for problems presented with graphs. When slope comparison problems were presented with equations, common incorrect strategies were "plugging in a number" for the x value of all three functions and comparing the results. For slope comparison problems presented with graphs, incorrect strategies were often some variation on "I just looked at the graph to see which line was highest." This approach indicates that students often experienced slope/height confusion (Leinhardt, Zaslavsky, & Stein, 1990). Conversely, 32% of participants reported a correct solution method or actually referenced the concept of slope when problems were presented with graphs relative to 18% when problems were presented with equations.

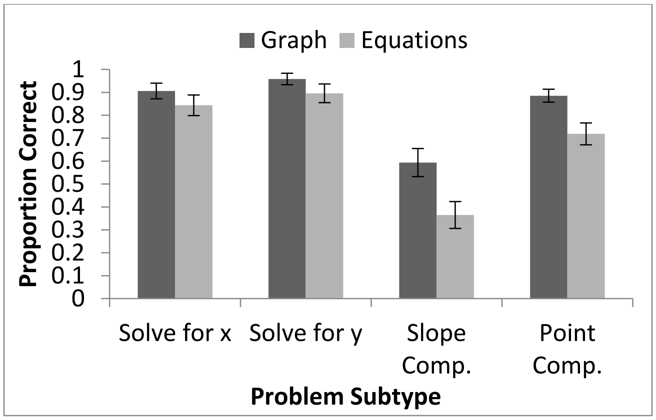


Figure 2: Mean proportion correct for the four problem subtypes presented in graphical and equation format. Error bars represent standard error.

Discussion

One striking result from Experiment 1 was the low overall accuracy on slope comparison problems in both formats, but particularly for slope comparison problems presented with equations. This pattern of results replicates the findings of Mielicki and Wiley (2016).

Participants' rephrasing responses suggest that graphs and equations may make different solution methods more accessible. Importantly, although the solution methods that students used when solving graphical slope problems sometimes led to a correct solution, these solution methods were not always indicative of conceptual understanding. For instance, it was possible for students to confuse slope with height when problems were presented with graphs (Leinhardt, Zaslavsky, & Stein, 1990) and still answer the problem correctly because the line with the highest height (*y* value) also had the highest slope.

However, even though performance was better on the slope comparison problems presented graphically, performance was still lowest overall on this problem type. It is possible that the overall low accuracy on slope comparison problems can be traced to difficulty during the problem comprehension phase. In previous work, it has been demonstrated that students often struggle with translating between stories, equations, and mathematical concepts (Cummins et al, 1988; Mayer, 1982; Nathan, Kintsch, & Young, 1992). In many contexts, students seem to fail to identify relevant mathematical principles. The results of Experiment 1 could be attributed to students not making the connection between the mathematical concept of slope and the demands of the task. If this is the case, then altering the language of the problem prompt to make the mathematical principles more clear should improve performance.

Experiment 2

The main goal of Experiment 2 was to test whether students would be able to solve equations requiring comparisons of slopes more successfully when the comprehension phase of problem solving is supported by presenting problem prompts with explicit mathematical terminology as opposed to natural language (as in Experiment 1).

This study focused specifically on only the two interpretation problem subtypes (point comparison, slope comparison) from Experiment 1. Manipulating the linguistic form of the problem prompt was not expected to have an effect on performance for point comparison problems. In Experiment 1, performance on point comparison problems benefitted from graphical format. This graphical advantage was consistent with the predictions from the cognitive task analysis that point comparison problems presented with equations were more computationally demanding than those presented with graphs. Thus the same pattern of results was expected for point comparison problems presented with mathematical terminology in Experiment 2.

If the graphical advantage for slope comparison problems found in Experiment 1 was due to graphs making different (though not necessarily correct) strategies accessible to participants, then supporting the comprehension phase should eliminate this advantage because participants should rely less on incorrect strategies if they can better access the underlying mathematical concept of slope. Thus, it was predicted that the graphical advantage found in Experiment 1 would be replicated when problems were presented with natural language, but that this advantage would be eliminated when problems were presented with explicit mathematical terminology. In addition, an overall main effect of linguistic form was expected because problem performance should improve overall when the comprehension phase is supported.

Method

Participants

A sample of 32 (20 female) undergraduate students from the University of Illinois at Chicago participated in exchange for course credit. Participants were mostly first year undergraduate students (ages ranging from 17 to 29), and had taken 0 to 3 math courses since starting college. Many participants (84%) reported a science (Biology, Chemistry, Psychology, Neuroscience) or health-related (Pre-dental, Pre-nursing, Kinesiology, Occupational Therapy) major. No participants reported pursuing a mathematics major.

Materials

A subset of slope comparison and point comparison problems from Experiment 1 was used for Experiment 2, and a second version of each problem was created in which the problem prompts were rephrased using explicit mathematical terminology instead of natural language. For example, prompts like "which cab company charges the most per mile?" were changed to "which line has the highest slope?" Each participant completed 8 problems, which were divided evenly between format (graphs, equations), type (point comparison, slope comparison), and linguistic form (natural language, mathematical terminology). These were the main manipulations of interest.

In addition, several presentation features were counterbalanced across participants. Items were blocked by linguistic form, and the order of the blocks was counterbalanced across participants with half completing natural language problems first and half completing mathematical terminology problems first. Within each block, the first and fourth problems were presented with one representation, and the second and third problems were presented with the other representation. Slope comparison and point comparison problems were alternated within each block. The order of problem type presentation and the order of format presentation were counterbalanced across participants. This 2x2x2 design resulted in 8 versions of the task, and 4 participants completed each version.

Procedure

The procedure was the same as the procedure in Experiment 1 except that rephrasing responses were not collected.

Results

A 2x2x2 within-subjects ANOVA was conducted with format (graph, equation); problem type (slope comparison, point comparison); and linguistic form (natural language, mathematical terminology) as independent variables and proportion correct as the dependent variable. As shown in Figure 4, the analysis revealed a main effect of linguistic form with problem prompts presented with mathematical terminology being solved more accurately than problem prompts presented with natural language, $F(1,31) = 8.03$, $p < .01$, $\eta_p^2 = .21$. There was also a main effect of format, with higher accuracy on problems presented with graphs than equations, $F(1,31) = 6.55$, $p < .05$, $\eta_p^2 = .17$. There was no main effect of problem type, $F < 1$. The linguistic form by representation and representation by problem type interactions were not significant, $F < 1$, and the three way interaction also did not reach significance, $F(1,31) = 2.99$, $p = .09$.

However, there was a linguistic form by problem type interaction, $F(1,31) = 7.79$, $p < .01$, $\eta_p^2 = .20$. Follow up paired-sample *t*-tests collapsing across representation revealed no difference in performance on point comparison problems based on linguistic form, $t < 1$; however, presenting problems in mathematical language significantly improved performance on slope comparison problems, $t(31) = 4.45$, $p < .001$.

Because it was predicted *a priori* that the results from Experiment 1 would be replicated for slope comparison problems presented in natural language form, and that mathematical language should eliminate the graphical advantage, paired-samples *t*-tests were conducted to compare performance on slope comparison problems presented with graphs with performance on problems presented with equations for each linguistic form. As predicted, performance was better on slope comparison problems presented with graphs than presented with equations when problems were in natural language form, $t(31) = 2.29$, $p < .05$. There was no difference in performance on slope comparison problems when presented in mathematical terminology, $t < 1$.

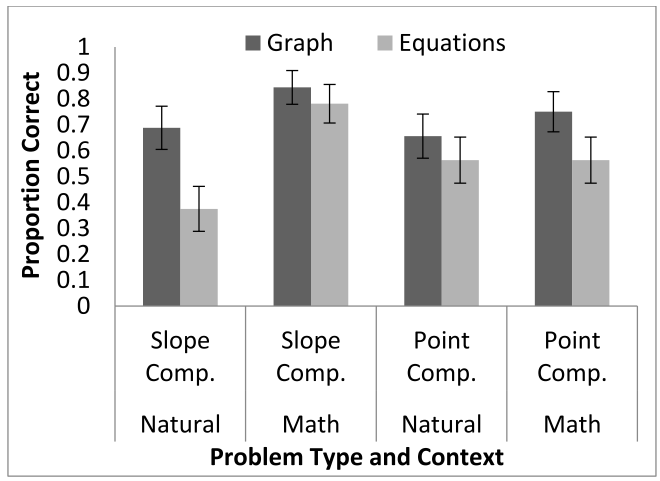


Figure 4: Mean proportion correct for slope comparison and point comparison problems presented in graph and equation format and in natural language or mathematical terminology form. Error bars represent standard error.

Discussion

The main result from Experiment 2 was that accuracy on slope comparison problems, particularly when presented with equations, improved when problems were presented in mathematical language. In addition, the graphical advantage found for slope comparison problems presented with natural language was eliminated when problems were presented with mathematical terminology.

General Discussion

The findings from Experiment 2 demonstrate that students are capable of solving slope comparison problems under certain conditions. Students do possess some understanding of the slope concept, but the limitations of this understanding are brought to light when problems are presented with natural language and particularly in equation format.

The graphical advantage for slope comparison problems found in Experiment 1 could be attributed to graphical format facilitating solution methods that lead to a correct answer without engaging the appropriate mathematical concept.

Student responses to the rephrasing prompts provide some support for this interpretation, but future research will need to address this possibility further by collecting trace data using think-aloud or eye-tracking methodology in order to better understand the ways that students are solving slope comparison problems in both formats.

Taken together, these results suggest that many undergraduates have not made the shift from procedural to conceptual understanding of slope. Students' difficulty recognizing or accessing the slope concept during problem solving highlights specific weaknesses in undergraduates' algebraic understanding. Slope has been widely acknowledged as a fundamental mathematical concept, with important implications for achievement in mathematics. Understanding of slope has been identified as central to success in precalculus and calculus (Carlson, Oehrtman, & Engelke, 2010), which in turn are required for many STEM career paths. Although slope is an important mathematical concept, it is also a notoriously difficult one for students to understand (Stump, 2001). Because linear equations are foundational, understanding and addressing weaknesses in students' conceptions of functions and slope represents an important step towards mending the leaky STEM pipeline.

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